The Amplitude Response of a Coupled Transmission Line, All-Pass Network Having Loss

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Abstract—**The voltage transfer function of a first- and second-order coupled transmission line, all-pass network having loss is derived. The amplitude response is calculated from the transfer function. It is shown that the amplitude response is completely determined when α, the real part of propagation function, is known. A method for calculating α is presented and an example is given. The results show that the losses (dielectric and conductor) cause periodic dips in the amplitude response of the all-pass networks. However, for practical materials and configurations, the peak amplitude loss is less than 0.4 dB for the first two periodicities. The results show that, for most applications, it is possible to cascade many first-order or second-order lines before the amplitude response must be equalized.**

I. Introduction

STEENAART [1] has analyzed nth order, coupled transmission line, all-pass networks. The proposed use of these lines was for wideband delay-equalizing networks. For this application, the lines were assumed lossless; thus the amplitude response was constant with frequency. For other applications, the line loss cannot be neglected. In particular, if a large number of lines are cascaded, the individual network loss is a significant parameter of the synthesis problem.

The purpose of this paper is to derive the transfer functions of a first-order and second-order all-pass line having small loss. The amplitude response can then be calculated from this transfer function.

The paper is organized as follows. Section II contains the basic definitions pertinent to the coupled transmission line, all-pass network. In Section III, the transfer function for the first- and second-order line is derived. The transfer function depends on α, the real part of the propagation constant; so the method by which α is calculated is discussed in Section IV. In Section V, the amplitude response is plotted for several practical first- and second-order lines.

II. General Considerations

The basic structure of the all-pass network consists of two parallel conductors located between ground planes and interconnected at one end. The structure has both vertical and horizontal symmetry and each conductor has a length, $L=\lambda_0/4$. A single unit of length, L, is called a first-order line, and is shown in Fig. 1. It is assumed that the structure

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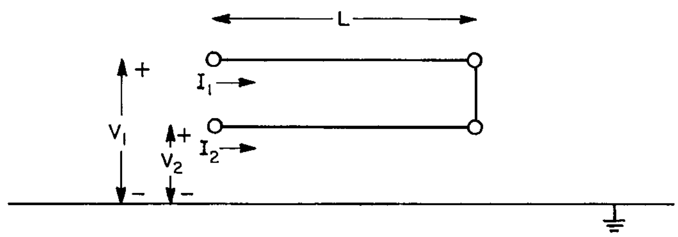


Fig. 1. First-order line.

is homogeneous and the losses are sufficiently small such that a TEM solution still exists on the line.

The even mode impedance of this structure is given by

$$Z_{oe} = \frac{1}{G_{11} + G_{12}} \tag{1a}$$

and the odd mode impedance is given by

$$Z_{oo} = \frac{1}{G_{11} - G_{12}} \tag{1b}$$

where G_{11} and G_{22} are the self susceptances per unit length of conductor 1 and conductor 2, respectively. G_{12} is the susceptance per unit length between conductors 1 and 2. The characteristic impedance of one conductor to ground is

$$Z_0 = (Z_{oe}Z_{oo})^{1/2} \tag{2a}$$

and the ratio of even-to-odd mode impedance is

$$\rho = Z_{oe}/Z_{oo} = \frac{G_{11} - G_{12}}{G_{11} + G_{12}}\,. \tag{2b}$$

The quantity, ρ, is important because the transfer function of the all-pass network is a function of ρ.

An all-pass second-order line is illustrated in Fig. 2. By definition Section II precedes Section I [1].

For the second-order line it is assumed that:

1) Each section is homogeneous and symmetrical.
2) The length of each section is $\lambda_0/4$.
3) The characteristic impedance, Z_0, of each section is equal; $Z_{01}=Z_{02}$.
4) The ratio of even-to-odd mode impedance varies; $\rho_1 \neq \rho_2$.

In the next section, the transfer function for the first- and second-order lines are presented.

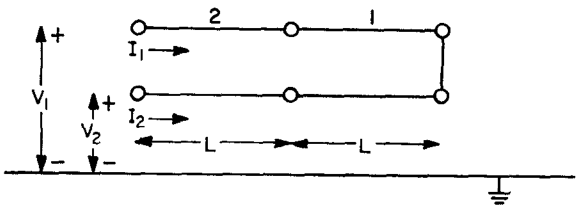


Fig. 2. Second-order line.

III. Transfer Function for First- and Second-Order Lines

The presence of loss on the lines causes the traveling wave to be attenuated by a factor proportional to αL (dissipative loss), and it causes the characteristic impedance of the line to become complex (mismatch loss). The attenuation constant consists of two parts, i.e., $\alpha=\alpha_c+\alpha_d$ where α_c is the conductor attenuation constant and α_d is the dielectric attenuation constant. In the following paragraphs, the lines are assumed to be in air, so $\alpha_d=0$. Later in the paper, a dielectric is introduced and the equations are modified accordingly.

The characteristic impedance in the presence of small conductor loss is given by

$$Z_0 = Z_0'(1 - j/Q)^{1/2} \tag{3}$$

where Z_0' = the characteristic impedance of the lossless network and $Q=\beta/2\alpha_c$. The susceptances are perturbed in the same manner, i.e.,

$$G_{12} = G_{12}'(1 - j/Q)^{-1/2} \tag{4a}$$

$$G_{22} = G_{11} = G_{11}'(1 - j/Q)^{-1/2}. \tag{4b}$$

The mismatch loss, represented by (3) and (4), can be shown to be small compared to the dissipative loss and will be neglected.

The voltage transfer function for the lossless first-order line is given by eq. (8) of Steenaart [1], as

$$\frac{V_2}{V_1} = \frac{\sqrt{\rho} - j\tan\theta}{\sqrt{\rho} + j\tan\theta} \tag{5}$$

where $\theta=\beta L$.

For the lossy case, substitution for θ of

$$\beta L - j\alpha_c L = \theta - j\psi \tag{6}$$

and expanding yields

$$\frac{V_2}{V_1} = \frac{(\sqrt{\rho} - \tanh\psi) - j\tan\theta(1 - \sqrt{\rho}\tanh\psi)}{(\sqrt{\rho} + \tanh\psi) + j\tan\theta(1 + \sqrt{\rho}\tanh\psi)} \tag{7}$$

which is an expression for the voltage transfer ratio of the lossy first-order line.

Now ψ is $\ll 1$ (typically on the order of 0.001). Thus, $\tanh\psi \approx \psi$. Then

$$\left|\frac{V_2}{V_1}\right| \approx \sqrt{\frac{\left(1 - \frac{\psi}{\sqrt{\rho}}\right)^2 + \frac{\tan^2\theta}{\rho}(1 - \psi\sqrt{\rho})^2}{\left(1 + \frac{\psi}{\sqrt{\rho}}\right)^2 + \frac{\tan^2\theta}{\rho}(1 + \psi\sqrt{\rho})^2}}. \tag{8}$$

Again, since ψ is small, $[1-(\psi/\sqrt{\rho})]^2 \approx 1-2\psi/\sqrt{\rho}$ and $(1-\psi\sqrt{\rho})^2 \approx 1-2\psi\sqrt{\rho}$. Thus,

$$\left|\frac{V_2}{V_1}\right| \approx \sqrt{\frac{1 - \frac{2\psi\sqrt{\rho}(1 + \tan^2\theta)}{\rho + \tan^2\theta}}{1 + \frac{2\psi\sqrt{\rho}(1 + \tan^2\theta)}{\rho + \tan^2\theta}}}$$

$$= \sqrt{\frac{1 - \Gamma}{1 + \Gamma}}. \tag{9}$$

Now

$$\frac{1 - \Gamma}{1 + \Gamma} \approx 1 - 2\Gamma \qquad \text{for small } \Gamma$$

and

$$\sqrt{1 - 2\Gamma} \approx 1 - \Gamma \qquad \text{for small } \Gamma.$$

Therefore,

$$\left|\frac{V_2}{V_1}\right| \approx 1 - \frac{2\psi\sqrt{\rho}(1 + \tan^2\theta)}{\rho + \tan^2\theta}, \tag{10}$$

which is a general expression for any θ.

When θ is an odd multiple of $\pi/2$,

$$\left|\frac{V_2}{V_1}\right| \approx 1 - 2\psi\sqrt{\rho}. \tag{11a}$$

When θ is an even multiple of $\pi/2$,

$$\left|\frac{V_2}{V_1}\right| \approx 1 - \frac{2\psi}{\sqrt{\rho}}. \tag{11b}$$

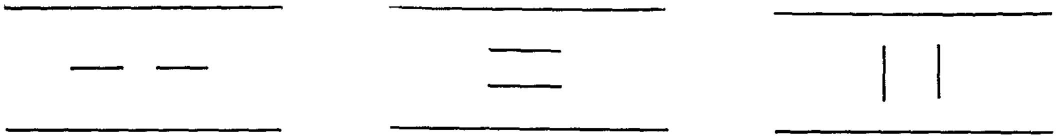
The lossy voltage transfer function for the second-order line is obtained by following the procedure used in Appendix I of Steenaart [1] after substitution for θ of

$$\theta_1' = \theta - j\psi_1 = \beta L - j\alpha_{c_1}L \tag{12a}$$

$$\theta_2' = \theta - j\psi_2 = \beta L - j\alpha_{c_2}L. \tag{12b}$$

The result is, for the second-order line,

$$\frac{V_2}{V_1} = \frac{\left[1 - \frac{\rho_2}{\rho_1}\tan\theta_1'\tan\theta_2'\right] - j\left[\frac{\tan\theta_1'}{\sqrt{\rho_1}} + \frac{\tan\theta_2'}{\sqrt{\rho_2}}\right]}{\left[1 - \frac{\rho_2}{\rho_1}\tan\theta_1'\tan\theta_2'\right] + j\left[\frac{\tan\theta_1'}{\sqrt{\rho_1}} + \frac{\tan\theta_2'}{\sqrt{\rho_2}}\right]}. \tag{13}$$



(a) (b) (c)

Fig. 3. Three geometrical configurations of coupled strip lines.

Substitution of θ = an even or odd multiple of $\pi/2$ in (13) and use of approximations similar to those for the first-order line yields

$$\left|\frac{V_2}{V_1}\right|_{\pi/2} \approx 1 - \frac{2(\psi_1\sqrt{\rho_1} + \psi_2\sqrt{\rho_2})}{\rho_2} \tag{14a}$$

$$\left|\frac{V_2}{V_1}\right|_{\pi} \approx 1 - 2\left(\frac{\psi_1}{\sqrt{\rho_1}} + \frac{\psi_2}{\sqrt{\rho_2}}\right). \tag{14b}$$

The transfer function for both the first- and second-order lines is a function of α, β, and ρ. It is assumed that β is given by

$$\beta = \omega(\mu_0\epsilon_0)^{1/2} \tag{15}$$

which is the same as in the lossless case. μ_0 and ϵ_0 are the free-space permeability and permittivity, respectively.

Thus, the problem of calculating the amplitude response becomes: Given a ρ (or ρ_1 and ρ_2 for the second-order line), calculate the variation of α_c with frequency.

The method by which α_c is calculated is presented in the next section; an example is also given.

IV. Calculation of α

The geometrical configuration of the conductors in the transmission line network can be either coplanar [Fig. 3(a)] or broadside-coupled [Fig. 3(b) and (c)]. Once the geometrical configuration is chosen, the even and odd mode impedances can be expressed as functions of the dimensions of the network [4]–[7]. These dimensions are b, the ground plane spacing, s, the spacing between the two conductors, and w, the width of each conductor. In practical applications, the conductors have finite thickness, t, which can be included in the expressions for the even and odd mode impedances [8]. All dimensions are in meters. Once the expressions for the even and odd mode impedances are obtained, α_c is calculated by the method presented by Cohn [9]. For the configurations shown in Fig. 3(b) where the conductors are in air, the attenuation constant for the all-pass network is given by

$$\alpha_c = \frac{R_s}{753.2}\left[\frac{1}{Z_{oe}}\left(\frac{\partial Z_{oe}}{\partial b} + \frac{\partial Z_{oe}}{\partial s} - \frac{\partial Z_{oe}}{\partial w} - \frac{dZ_{oe}}{\partial t}\right) + \frac{1}{Z_{oo}}\left(\frac{\partial Z_{oo}}{\partial b} + \frac{\partial Z_{oo}}{\partial s} - \frac{\partial Z_{oo}}{\partial w} - \frac{\partial Z_{oo}}{\partial t}\right)\right] \tag{16}$$

where R_s is the surface resistance of the conductor in ohms per meter, and α_c is in nepers per meter. Since R_s is proportional to the square root of frequency ($R_s = 8.04 \times 10^{-3}(f)^{1/2}$ for silver, where f is in gigahertz), α_c is proportional to $(f)^{1/2}$.

As an example of the application of (16), the configuration in Fig. 3(b) is chosen, and it is assumed that

$$\frac{w}{b-s} \geq 0.35. \tag{17}$$

Using the equations contained in Cohn [6], [8], the first term in the brackets of (16) is given by

$$\frac{1}{Z_{oe}}\left(\frac{\partial Z_{oe}}{\partial b} + \frac{\partial Z_{oe}}{\partial s} - \frac{\partial Z_{oe}}{\partial w} - \frac{\partial Z_{oe}}{\partial t}\right) = \frac{Z_{oe}}{188.3(d-s)}\left[1 + \frac{1}{\pi}\ln\left(\frac{d+2t}{s+2t}\right)\right] \tag{18}$$

where

$$d = b - 2t.$$

The second term in the brackets is given by

$$\frac{Z_{oo}}{188.3}\left\{\frac{w}{s^2} - \frac{\ln(d/s)}{\pi(d-s)} + \frac{(d-s)}{\pi s^2}\ln\left(\frac{d}{d-s}\right) + \frac{2}{\pi s}(1 + t/s)[\ln(1+t/s) - \ln(t/s)]\right\}. \tag{19}$$

Z_{oe} is given by

$$Z_{oe} = \frac{188.3}{\frac{w}{d-s} + 0.4413 + \frac{1}{\pi}\left[\ln\left(\frac{d+2t}{d-s}\right) + \left(\frac{s+2t}{d-s}\right)\ln\left(\frac{d+2t}{s+2t}\right)\right]} \tag{20}$$

and Z_{oo} is given by

$$Z_{oo} = \frac{188.3}{w\left[\frac{d}{s(d-s)}\right] + \frac{d}{\pi s}\left[\ln\left(\frac{d}{d-s}\right) + \left(\frac{s}{d-s}\right)\ln\left(\frac{d}{s}\right)\right] + \frac{2}{\pi}\left[\left(1+\frac{t}{s}\right)\ln\left(1+\frac{t}{s}\right) - \frac{t}{s}\ln\left(\frac{t}{s}\right)\right]}. \tag{21}$$

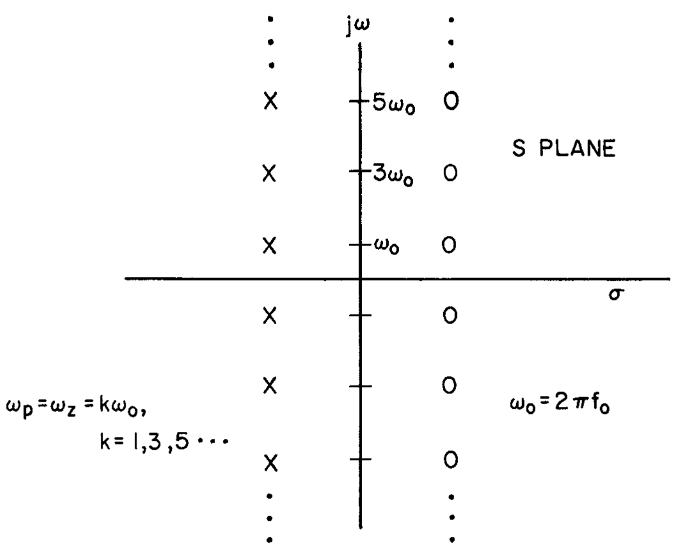


Fig. 4. Pole-zero plot of first-order line.

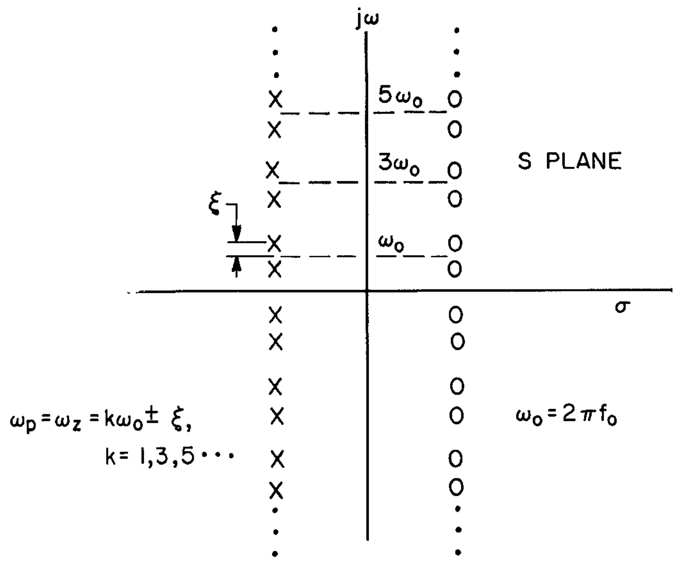


Fig. 5. Pole-zero plot of second-order line.

If the structure is filled with a homogeneous dielectric material with a relative dielectric constant of ϵ_r, the equations are modified as follows:

1) Equations (16), (18), and (19) are each multiplied by $(\epsilon_r)^{1/2}$. Equations (20) and (21) are each divided by $(\epsilon_r)^{1/2}$.
2) α_c is replaced by α in (6) and (12) where $\alpha=\alpha_c+\alpha_d$ and $\alpha_d=[\pi(\epsilon_r)^{1/2}/\lambda]\tan\delta$ nepers per meter. λ is the free space wavelength and $\tan\delta$ is the loss tangent of the dielectric.
3) Equation (15) is multiplied by $(\epsilon_r)^{1/2}$.

In the next section, the amplitude response of several first- and second-order lines are plotted for a specific configuration and ground plane spacing.

V. Amplitude Response Curves

Before referring to the amplitude response curves, it is instructive to study the pole-zero pattern of the first- and second-order lossless lines which are shown in Figs. 4 and 5, respectively [1].

The quantity f_0 is given by

$$f_0 = c/\lambda_0 \tag{22}$$

where c is the velocity of light in air.

The pole-zero pattern, in both cases, is periodic with frequency and the distance between the poles (zeros) and the imaginary axis is constant. Thus, the time delay of both the first- and second-order lines is periodic with frequency.

Now the peak loss of the amplitude response (in nepers) of an all-pass network is proportional to the frequency times the time delay and inversely proportional to Q [10].

$$\text{Peak Amplitude Loss (nepers)} \approx \frac{\omega \times \text{time delay}}{Q}. \tag{23}$$

Since the peak time delay of the networks is constant and periodic with frequency, and since Q is proportional to the square root of frequency, the amplitude response should be periodic with frequency, and the peak loss (in nepers) should increase with the square root of frequency. This means that once the peak loss for the lowest point is known, the magnitude of the peak loss for other amplitude minima can be obtained quite readily.

The amplitude response of several all-pass first- and second-order lines are plotted in Figs. 6 to 9. The following conditions were assumed:

1) The geometrical configuration was that shown in Fig. 3(b).
2) $b=0.75$ inch; $t=0.04$ inch; $\epsilon_r=1$.
3) Equation (16) is valid. [Given conditions 1) and 2), this means that $\rho \leq 14.7$].
4) The conductors were silver-plated.
5) The dielectric medium was air.

Figure 6 shows the amplitude response of a first-order line for $\rho=14$, $\rho=10$, and $\rho=5$, respectively. f_0 is 0.25 GHz. Figure 7 is the same as Fig. 6 except that $f_0=0.15$ GHz. The values of α_c ranged from 8×10^{-3} at $f_0=0.25$ GHz and $\rho=14$ to 1.6×10^{-3} at $f_0=0.15$ GHz and $\rho=5$.

The peak loss increases as ρ increases because the peak time delay is proportional to $(\rho)^{1/2}$, and also Q decreases as ρ increases. For a given ρ the peak loss decreases as f_0 is increased because αL, the attenuation per quarter wave length, decreases.

The amplitude response of several second-order lines is shown in Figs. 8 and 9. Again, the peak loss is proportional to the peak time delay, and the location of the peak loss is periodic with frequency. The second-order line with $\rho_1=2$, $\rho_2=14$ has the highest peak time delay, while the line with $\rho_1=10$, $\rho_2=5$ has the lowest peak time delay. The other lines have the same peak time delay, but the peaks occur at a different frequency [1]. If several lines have the same ρ_2, the

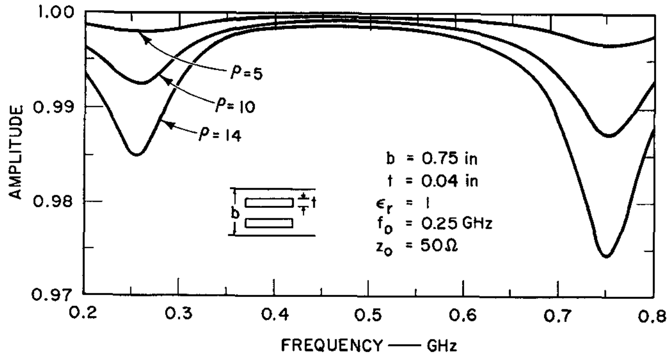


Fig. 6. Amplitude response of first-order lines in air for $f_0 = 0.25$ GHz.

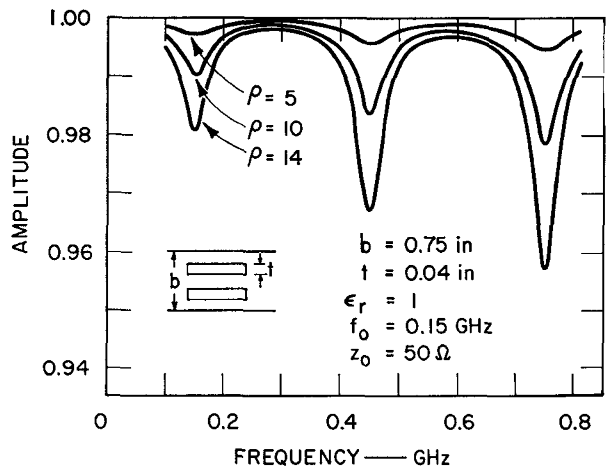


Fig. 7. Amplitude response of first-order lines in air for $f_0 = 0.15$ GHz.

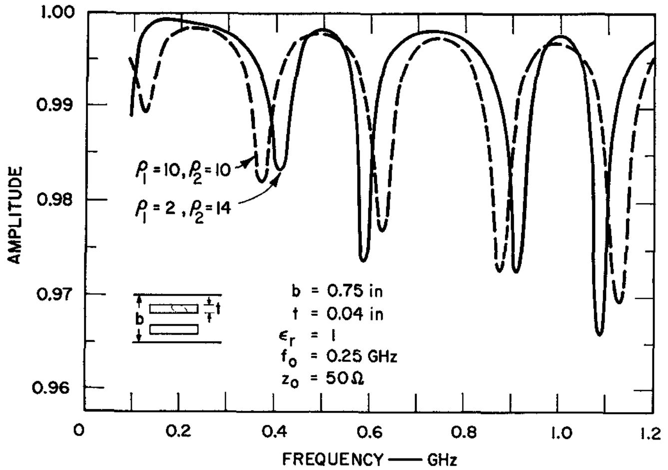


Fig. 8. Amplitude response of second-order lines in air for $f_0 = 0.25$ GHz.

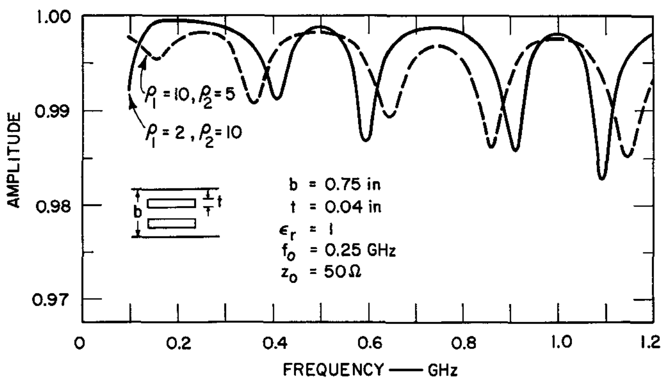


Fig. 9. Amplitude response of second-order lines in air for $f_0 = 0.25$ GHz.

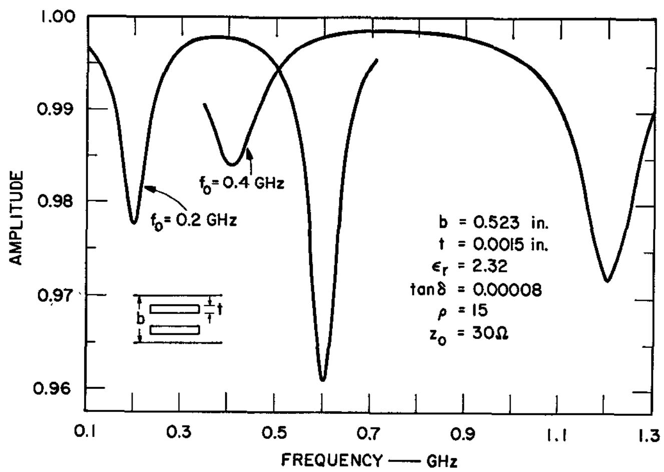


Fig. 10. Amplitude response of first-order lines in dielectric.

peak loss will be greatest in the line that has the highest ρ_1, because Q decreases with ρ.

Figure 10 is a plot of the amplitude response of two first-order lines with the same ρ but different f_0s when both structures are immersed in a dielectric medium. The physical parameters of the lines are given in the figure. The response of these lines are similar to the air dielectric lines except that the peak loss is slightly greater (as is expected) when there is dielectric present. The value of α_c is 0.014 at $f_0=0.4$ GHz.

Some of the above lines were built, and the amplitude measurements made on these lines were all within twelve percent of the calculated values.

VI. Conclusion

The results show (Figs. 6 to 10) that the losses cause a dip in the amplitude response of the coupled transmission line, all-pass networks just as they do for the all-pass network composed of lumped elements. However, they also show that for practical configurations and materials, the peak amplitude loss is small for the first two periodicities. Thus, it is possible to cascade many first-order or second-order lines before the amplitude response must be equalized (for most applications).

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Propagation Through a Twisted Medium

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Abstract—**An explicit solution is obtained for propagation through a uniform twisted anisotropic medium subject to the conditions that propagation is along the twist axis, and that the structure is fine. The propagation constants are altered and coupling exists between the propagation modes. A parameter is defined which indicates the tendency of the radiation to adhere to the structure of the medium. The effects at boundary discontinuities are discussed, and tapers to an isotropic medium are dealt with. The particular application of the theory to cases of polarization conversion, circular to plane, and plane to plane are discussed.**

I. Introduction

DURING the course of development of a plastic strip polarizer for an antenna fed by a line source, it became apparent that the inclination of the strips was not a simple parameter which could be simply related to the plane of polarization. The finite thickness and cylindrical shape of the polarizer combine to require individual strips of helicoidal shape. The inclination angle of the strips varies significantly about the nominal 45°; and on looking through the polarizer, the strip structure appears to twist about the propagation axis.

Consequently, the problem of propagation through a twisted medium became of interest. Specifically, it was necessary to determine the correct strip inclination and the effect of the twist on the differential phase shift. The problem was idealized by assuming an infinitesimally fine structure, a uniform twist, and normal incidence.

Propagation through a twisted medium is a special case of a more general theory developed by Suchy.[1] Recently, a paper by van Doeren[2] has treated the twisted medium problem by means of direct computer solutions of the differential equations. The solution in explicit form for the case of a uniform twist is given in the following.

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